AMG-NYU Memorandum No. 3 AMP Memorandum No. 38.2

REPEATED REFLECTION OF A SHOCK AGAINST A RIGID WALL

by Harvey Cohn

A MEMORANDUM SUBMITTED
by the
APPLIED MATHEMATICS GROUP, NEW YORK UNIVERSITY
to the
APPLIED MATHEMATICS PANEL
NATIONAL DEFENSE RESEARCH COMMITTEE

April, 1943

RESTRICTED

Copy No. 60

REPEATED REFLECTION OF A SHOCK AGAINST A RIGID WALL

Harvey Cohn

Let a piston begin to move with constant velocity w (> 0), in (say) the $+x$ direction, into a cylinder filled with gas originally at rest. We assume that the piston drives a shock* $(x = \xi(t))$ through the cylinder, stirring the gas particles into motion with speed w, which implies, of course that $\dot{\xi} > w$. This shock causes a higher than atmospheric pressure to react on the piston. We shall call this shock the <u>initial incident shock</u>, and the reaction against the piston, the <u>initial incident pressure</u>, p_{0*}.

Let us suppose, now, that the cylinder is closed by a wall. Then, after the initial incident shock reaches the wall, an <u>initial reflected shock</u> is assumed to originate at the wall and travel backwards in order to nullify the forward particle motion produced by the initial incident shock. The new shock will leave in its wake a column of gas <u>at rest</u> but at a still higher pressure, namely the <u>initial reflected pressure</u>, p_1, that will react on the wall.

Soon the initial reflected shock will recede far enough to meet the piston. At this instant, the piston will be moving forward (with speed w) into the column of gas, which is again at rest although at the high initial reflected pressure. This situation, like the original one, gives rise to a <u>first incident shock</u> which is followed in turn by a <u>first reflected shock</u>, the pressure on the piston or on the wall increasing at each stage. Thus the

* It may be seen, from (I), below, that each of the shocks has a constant speed, so that $\dot{\xi}$ is a (different) constant for each.

x t plane will be divided into triangular regions, each with constant pressure, density, and particle speed (0 or w):

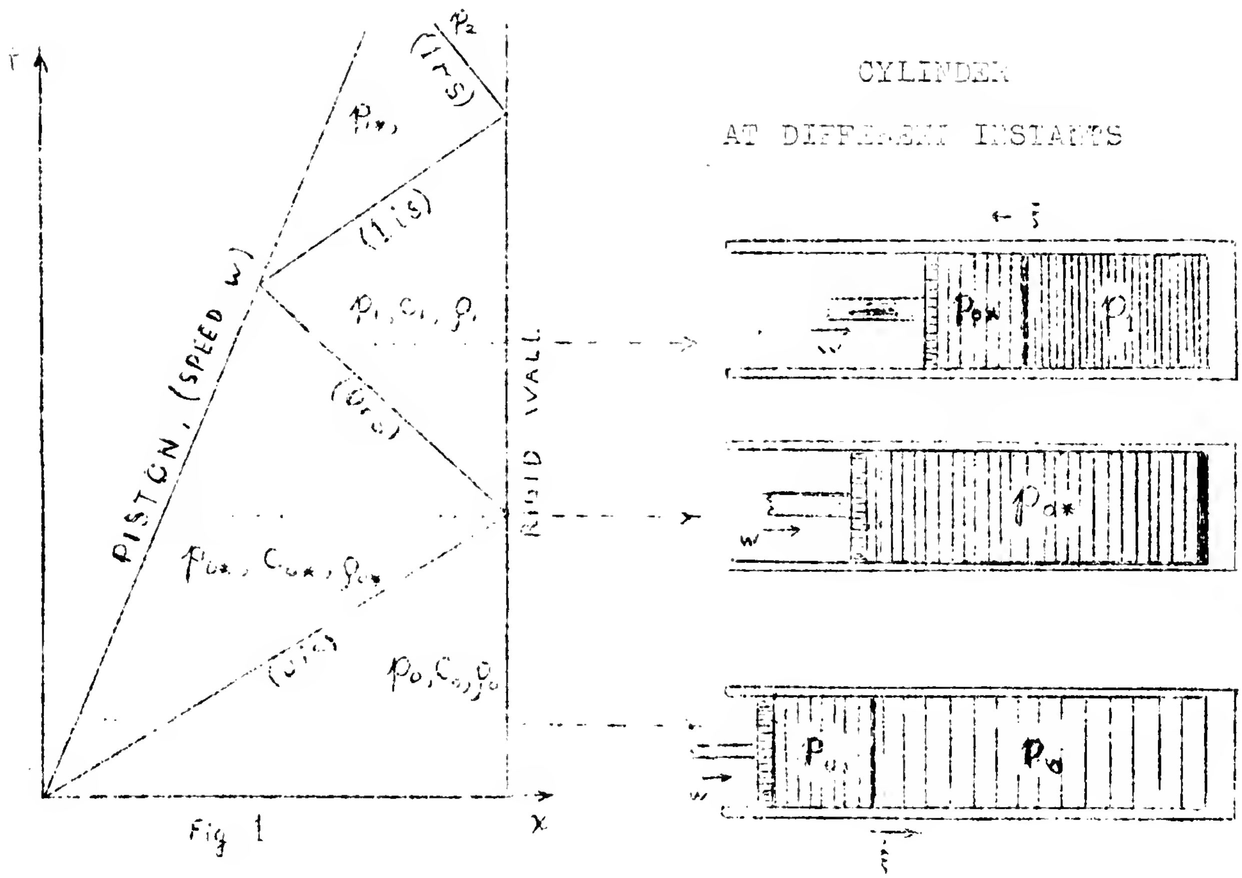


Fig 1

SYMBOLS:

SHOCKS:

- (0is) initial incident shock
- (0rs) initial reflected shock
- (1is) first incident shock
- (1rs) first reflected shock

which are followed by (2is) , (2rs) , &c.

PRESSURES:

- p_0 original (atmospheric) pressure
- p_{0*} initial incident pressure
- p_1 initial reflected pressure
- p_{1*} first incident pressure
- p_2 first reflected pressure, &c.

— — —

The object of this report is to find the <u>first</u>, <u>second</u>, ... <u>reflected pressures</u> p_1, p_2, ..., against the wall, for various piston speeds v.

<u>Discontinuity Conditions</u> *

The state of gas in cylinder is characterized by the variables:

p = pressure

ρ = density

u = particle velocity

We let γ denote the <u>adiabatic constant</u>, and we introduce the constant : $\kappa = \frac{\gamma - 1}{2}$. In air, $\gamma = 1.4$ $\kappa = .2$.

We shall also use the <u>sound velocity</u> , given by $c = \sqrt{\gamma p/\rho}$.

To find the successive pressures we use the Rankine-Hugoniot Discontinuity Conditions , which for our purposes may be taken in the following form:

Denote the two sides of a shock by A, B, and we consider all quantities pertaining to side B to be <u>known</u>. (Either the shock changes particles from state A (p_A, ρ_A, u_A) to state B (p_B, ρ_B, u_B) or vice versa.) If we let $x = \xi(t)$ describe the motion of the shock, we may express side A conveniently in terms of side B by finding, somehow,, a parameter R called the <u>shock index</u>: $R = (\dot{\xi} - u_B)/c_B$. In fact we have:

$$
(\mathrm{I}) \quad \begin{cases}
p_A/p_B = \frac{1}{\kappa+1}\left((2\kappa+1)R^2 - \kappa\right) \\
\rho_A/\rho_B = (\kappa+1)/\left(\kappa + \frac{1}{R^2}\right) \\
c_A/c_B = \frac{1}{\kappa+1}\sqrt{\left((2\kappa+1)R^2 - \kappa\right)\left(\kappa + \frac{1}{R^2}\right)} \\
\frac{u_A - u_B}{c_B} = \frac{1}{\kappa+1}\left(R - \frac{1}{R}\right)
\end{cases}
$$

RESTRICTED

---

* The Conditions used here will be further discussed in the forthcoming Manual prepared by the Appl. Math. Group at N.Y.U.

Thus our problem has two parts:

1. To find the shock indices for the transition from each of the triangles of Fig. 1 to its neighbor.

2. To find the successive pressures from the shock indices by means of the first formula in (I).

The λ - Sequences

We apply (I) across the initial incident shock (Oi*), identifying (A) in (I) with (O*) in Fig. 1 and
" (B) " " (O) " ".

We find for the last two ratios in (I):

$$(IIa)\quad \begin{cases} \dfrac{c_{o*}}{c_o} = \dfrac{1}{k+1}\sqrt{\left((2k+1)R^2 - k\right)\left(k + \dfrac{1}{R^2}\right)} \\ \dfrac{w-o}{c_o} = \dfrac{1}{k+1}\left(R - \dfrac{1}{R}\right) \end{cases}$$

If we introduce the dimensionless variable λ:

$$\lambda_{n*} = w/c_n, \quad \lambda_n = w/c_{n*},$$

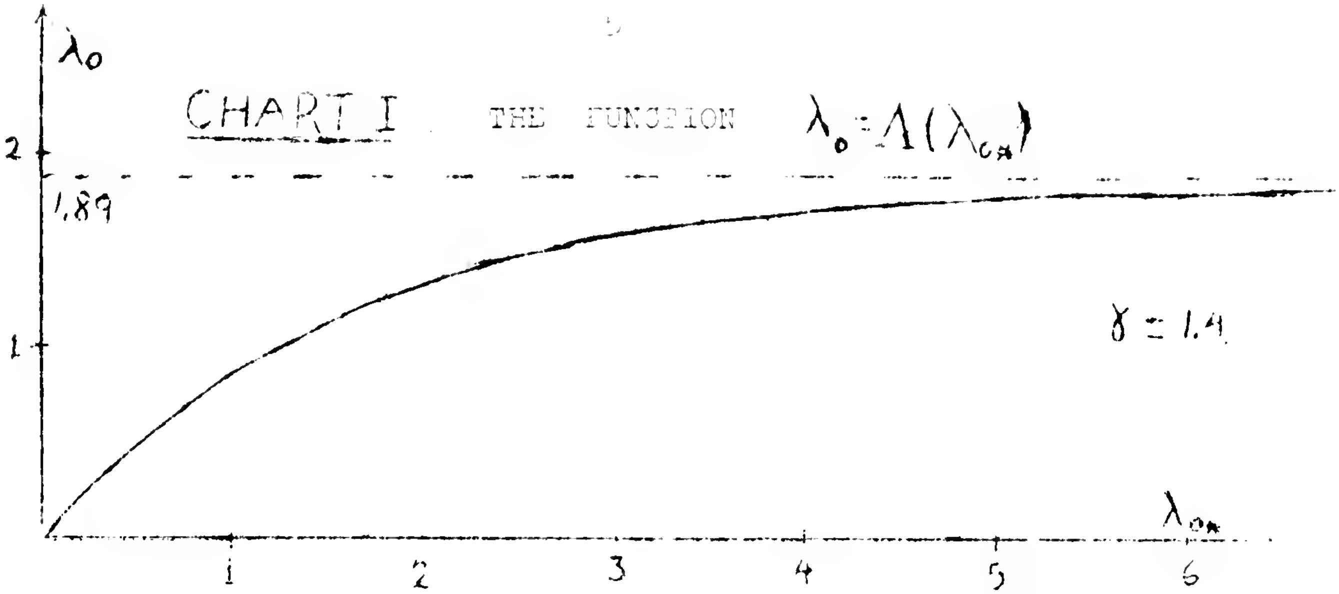
we find that (IIa) becomes:

$$(II)\quad \begin{cases} \lambda_{o*} = \dfrac{1}{1+k}\left(R - \dfrac{1}{R}\right) \\ \lambda_o = \left(R - \dfrac{1}{R}\right) \Big/ \sqrt{\left((2k+1)R^2 - k\right)\left(k + \dfrac{1}{R^2}\right)} \end{cases}$$

Now (II) represents a curve in the $(\lambda_o, \lambda_{o*})$ plane as R varies from 1 to ∞; we call this curve $\lambda_o = \Lambda(\lambda_{o*})$, it is plotted in Chart (I), below. We notice that this function has an asymptote at $\lambda_o = 1/\sqrt{k(2k+1)} = \delta_o$. For $k = .2$, $\delta_o = 1.89$. Thus as $\lambda_{o*}\ (= w/c_o) \to \infty$, $\lambda_o\ (= w/c_{o*}) \to \delta_o$.

From considerations of symmetry, we conclude that (II) holds across any shock, or

$$(III)\quad \begin{cases} \lambda_n = \Lambda(\lambda_{n*}) \\ \lambda_{(n+1)*} = \Lambda(\lambda_n) \end{cases} \quad n = 0, 1, 2, \ldots$$



(SEE (II) , p.4)

Hence, according to (III) p.4, for any given w or λ_{0*}, we may calculate a decreasing sequence of values of λ by using Chart I, above.

<u>Example</u>: Let w = 4.4 c_0, or let the piston that is 4.4 times supersonic.

$\lambda_{0*} = 4.40$	$\lambda_{1*} = 1.22$	$\lambda_{2*} = .85$	$\lambda_{3*} = .68$
$\lambda_0 = 1.72$	$\lambda_1 = .97$	$\lambda_2 = .74$	$\lambda_3 = .6$[illegible]

<u>Example</u>: Let w >> c_0, or let the piston be extremely supersonic; then our limiting λ sequence is

$\delta_{0*} = \infty$	$\delta_{1*} = 1.27$	$\delta_{2*} = .$[illegible]	$\delta_{3*} = .68$
$\delta_0 = 1.89$	$\delta_1 = .99$	$\delta_2 = .74$	$\delta_3 = .6$[illegible]

From the λ sequence, we determine each value of c . This sequence, however, is even more useful, for from it we shall find the shock-indices and pressure ratios.

Consider, for example, the particles with the initial incident pressure p_{0*} . These particles have speed w and they occupy a triangular portion of the x-t plane bounded by two shocks, (Oi*) and (Or*), see Fig. 1.

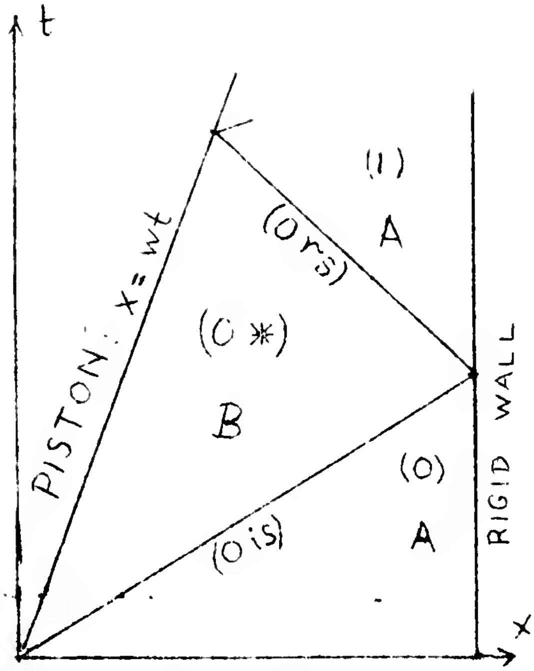


Fig. 1 bis

We shall apply the Shock Conditions by identifying the known state B with the state (0*). We then ask what two states <u>at rest</u> (0) and (1), each identified with A, determine (0is) the forward shock producing B and (0rs) the backward shock produced by B. (See Fig.1 bis.) In fact, the two states A correspond to the two roots (IVb) of a quadratic equation which results if we set up the Rankine - Hugoniot Conditions (I) to apply to some state A to the right of B and at rest.

Thus for some shock index R (that applies to the transition from B to either one of the A),

$$(\text{IV}) \qquad \frac{u_A - u_B}{c_B} \quad \text{or} \quad -\frac{w}{c_{0*}} = \frac{1}{K+1}\left(R - \frac{1}{R}\right)$$

The two values of R that are determined from

$$(\text{IV}\,a) \qquad -\lambda_0 = \frac{1}{K+1}\left(R - \frac{1}{R}\right)$$

are conjugate surds R_0, R_1 which we substitute into:

$$(\text{V}a) \qquad \frac{p_A}{p_B} \quad \text{or} \quad \frac{p_i}{p_{0*}} = \frac{(2K+1)R_i^2 - K}{K+1} \qquad (i = 0, 1)$$

Then the ratio p_1/p_0 may be determined as $p_1/p_{0*} : p_0/p_{0*}$ or, in other words, <u>the ratio p_1/p_0 depends only on</u> λ_0.

In fact, from (IV),

$$(\text{IV}b) \qquad R_i = -\frac{(1+K)}{2}\lambda_0 \mp \sqrt{\frac{(1+K)^2}{4}\lambda_0^2 + 1}, \qquad (i = 0, 1)$$

From (Va),

$$\frac{p_i}{p_0} = 1 + \frac{(2K+1)(K+1)\lambda_0^2}{2} \pm (2K+1)\lambda_0\sqrt{\frac{(1+K)^2}{4}\lambda_0^2 + 1} = g_{\pm}(\lambda_0).$$

RESTRICTED

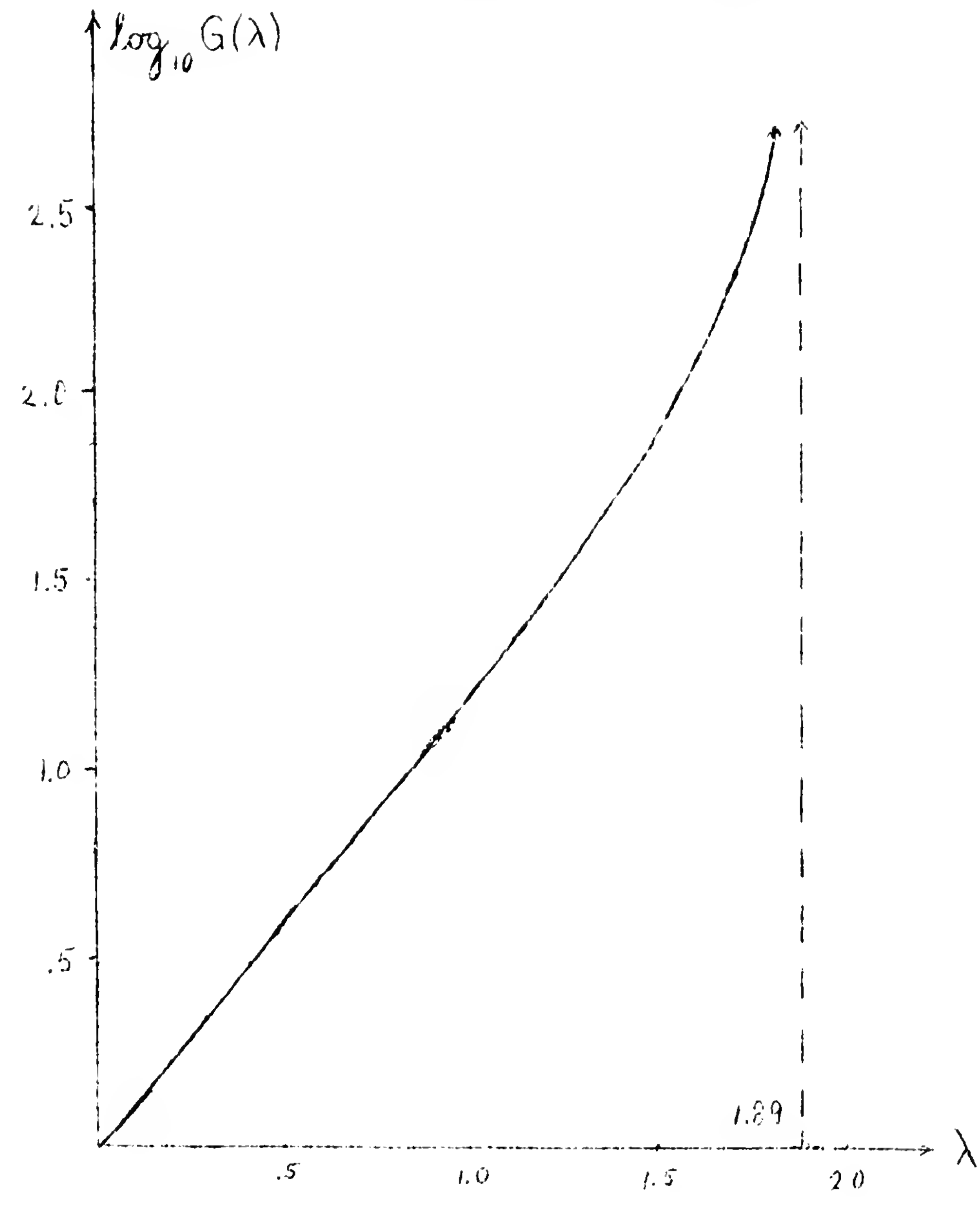
Finally, with [illegible] regard to choice of (±) signs, we have,

$$(\mathrm{V})\quad p_1/p_0 = \underline{q_+(\lambda_0)/q_-(\lambda_0) \quad \text{or} \quad G(\lambda_0)} \qquad \ldots)$$

or, more generally,

$$(\mathrm{V}\,b)\quad \underline{p_{n+1}/p_n = G(\lambda_n)}\,, \qquad p_{(n+1)*}/p_{n*} = G(\lambda_{(n+1)*}).$$

CHART II: $\log G(\lambda)$



($G(\lambda)$ is defined by (V).)

The function $\log_{10} G(\lambda)$ is plotted on Chart II, above, for values of from 0 to (almost) δ_0, (the only values of λ_0 that are possible for a given λ_{0*}). Since

$$p_n/p_0 = p_n/p_{n-1} \cdot p_{n-1}/p_{n-2} \cdots p_1/p_0 = G(\lambda_{n-1})G(\lambda_{n-2}) \cdots G(\lambda_0),$$

we may find $\log_{10} p_n/p_0$ by reading $\sum_{m=0}^{n-1} \log_{10} G(\lambda_m)$ off Chart II. By this method the <u>reflected pressures</u> p_n have been plotted on Chart III, below, for the range of w up to $9c_0$ and for $n = 1, 2, 3, 4$.

The values p_{0*}, of the initial incident pressure are plotted on Chart III for values of $\lambda_{0*} = w/c_0$, since in practice the initial incident shock might be characterized by its pressure rather than by its particle speed w. The curve of p_{0*} as a function of λ_{0*} is given (parametrically) by the first and last formulae in (I), p.3 :

$$(\text{VI}) \begin{cases} \lambda_{0*} \text{ or } w/c_0 = \dfrac{1}{\kappa+1}\left(R - \dfrac{1}{R}\right) \\ p_{0*}/p_0 = \dfrac{2\kappa+1}{\kappa+1}R^2 - \dfrac{\kappa}{\kappa+1} \end{cases}$$

where R varies from 1 to infinity. Now, the pressures p_{n*} can be found, if necessary, by means of

$$\log_{10} p_{n*}/p_0 = \sum_{m=1}^{n} \log_{10} G(\lambda_{m*}) + \log_{10} p_{0*}/p_0$$

<u>Intense Shocks</u>

The purpose of Chart III is primarily to give the <u>orders of magnitude</u> of the pressures p_n, $n = 1,2,3,4$, for $w/c_0 < 9$ or for $p_{0*} < 130$ atm. For <u>initial incident shocks</u> of greater strength we use the previous formulae asymptotically. For example with $w/c_0 = \lambda_{0*}$ large, from equations (II),

$$\lambda_0 \approx \delta_0 - (3\kappa+1)\sqrt{\kappa(2\kappa+1)}/2(\kappa+1)\lambda_{0*}^2$$

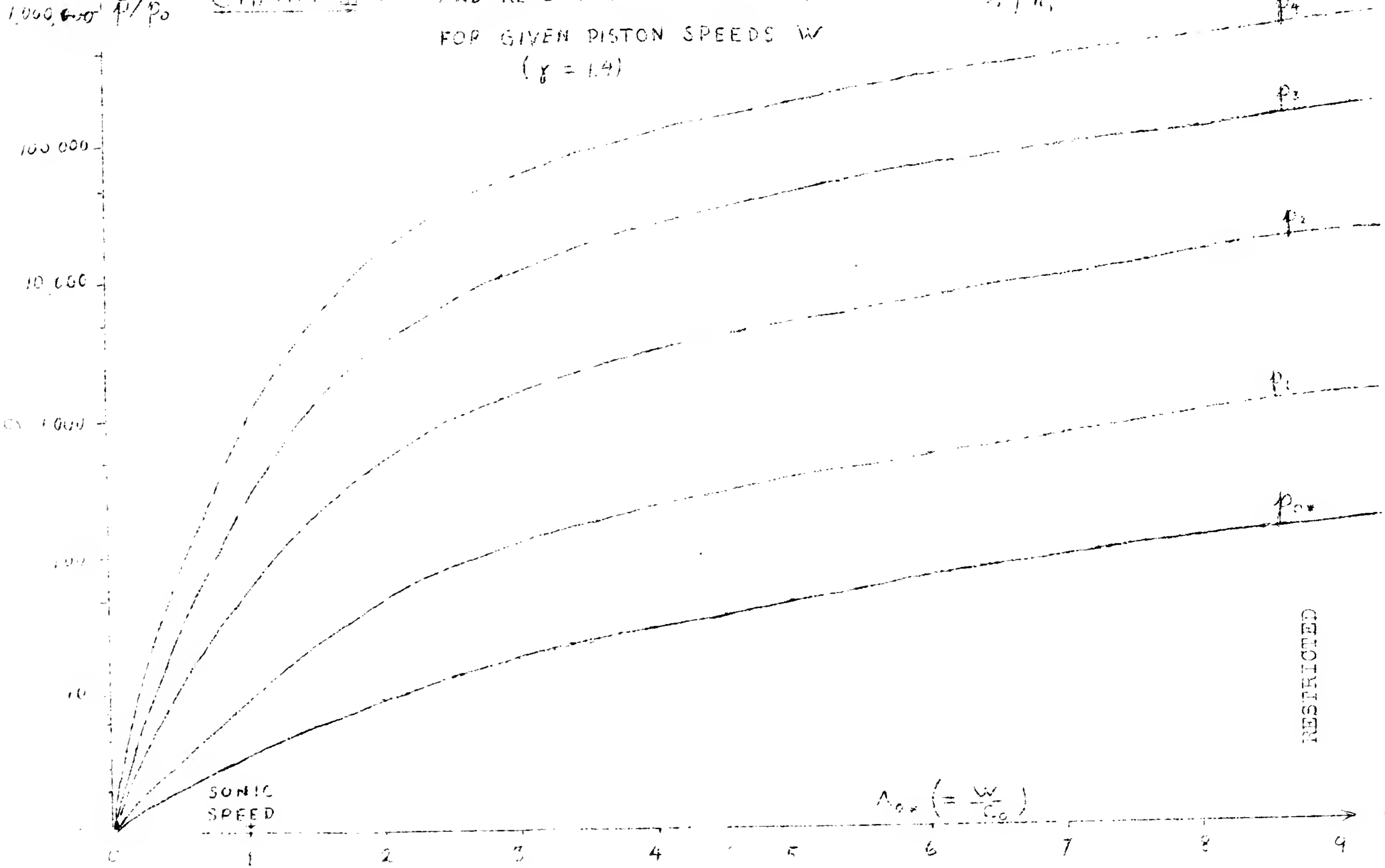
and

$$\frac{p_1}{p_0} = G(\lambda_0) \approx \left\{ \lambda_{0*}^2 (2\kappa+1)(\kappa+1) \right\} \frac{3\kappa+1}{\kappa}$$

RESTRICTED

CHART II

INITIAL INCIDENT PRESSURE (AGAINST PISTON), p_{0*},
AND REFLECTED PRESSURES (AGAINST WALL), p_n,
FOR GIVEN PISTON SPEEDS W
$(\gamma = 1.4)$



But from (VI), the quantity in braces $\{ - - - - \}$ is recognized as p_{0*}/p_0 for λ_{0*} large, or $w \gg c_0$,

$$p_1/p_0 = G(\lambda_0) \approx \frac{3\kappa+1}{\kappa}\frac{p_{0*}}{p_0} = 8\, p_{0*}/p_0 \quad (\text{for } \kappa = .2);$$

and for p_{0*} large, $p_1 \approx 8p_{0*}$, while the values of λ are the $\mathcal{E}$ sequence on p.5, above. We find, thus:

$$p_2/p_1 \approx G(.99) = 17\ , \quad p_3/p_2 \approx G(.74) = 8.1, \quad p_4/p_3 \approx G(.63) = 6$$

Or, using these factors cumulatively, we find, roughly:

THE REFLECTED PRESSURES

$$\left.\begin{aligned} p_1 &= 8p_{0*} \\ p_2 &= 135p_{0*} \\ p_3 &= 1100p_{0*} \\ p_4 &= 6500p_{0*} \end{aligned}\right\}$$

where p_{0*}, the <u>initial incident pressure</u>, is large

<u>Effect of an Indefinite Number of Reflections</u>

Let us assume, as a mathematical problem primarily, that an infinite number of reflections occur.

We then consider the sequence $\lambda_1, \lambda_2, \ldots$. It decreases steadily to zero and its behavior is significant; for in formulae

(IV, V): $\quad R_i = \mp 1 + \dfrac{(1+\kappa)}{2}\lambda_n + \text{terms in } \lambda_n^2\ ,$

$$\therefore\quad p_{n+1}/p_n = 1 + 2(2\kappa+1)\lambda_n + \text{"} \quad \text{"} \quad \text{"}, \quad (\gamma = 2\kappa+1)$$

$$\rho_{n+1}/\rho_n = 1 + 2\lambda_n + \text{"} \quad \text{"} \quad \text{"},$$

Now it can be shown, although the proof is omitted, that as n becomes very large or as λ_n becomes very small,

$$\lambda_n \approx \frac{1}{2\kappa n}\ ; \quad \text{i.e.} \quad c_n \approx 2\kappa n w,$$

hence $\sum_1^\infty \lambda_n$ diverges while $\sum_1^\infty \lambda_n^2$ converges. This leads to the following interesting conclusions:

(1) As the number of shocks increases, the pressure of the gas between the piston and the wall increases beyond all limit while the volume of the gas shrinks to zero.

(2) We may conclude from the discontinuity conditions (I) that if R is close to ± 1, (or if the shock is weak), the entropy changes are of <u>third</u> order compared with the pressure or density changes. (Entropy $=$ const. $\cdot \log p\rho^{-\gamma}$.) Hence the increments of entropy due to each shock converge or

$$p_n \rho_n^{-\gamma} \rightarrow \text{limit} \quad \text{as } n \rightarrow \infty$$

The limiting value of the entropy, however, must steadily increase beyond limit as $w/c_0 = \lambda_{0*}$ increases.

Date Due
